PERMANENT ROTATIONS OF A HEAVY GYROSTAT HAVING A STATIONARY POINT

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Kharlamov [1] and Tsodokova [2] have specified the cone of permanent axes of a gyrostat having a single stationary point. The present paper is an investigation of this cone and the domain of stability of the permanent rotations. It is appropriate at this point to note that Kharlamov's remarks in [1] concerning the present author's papers [3] and [4] are quite valid.

Our notation is as follows: $Ox_1x_2x_3$ is a coordinate system invariably connected with the solid portion of the gyrostat. The origin of the system lies at the point O of the gyrostat and its axes coincide with the principal axes of inertia. A_1, A_2, A_3 are the principal moments of inertia: I is the inertia tensor of the gyrostat for the stationary point O: $\mathbf{k}(k_1, k_2, k_3)$ is the gyrostatic moment: $\mathbf{r}(r_1, r_2, r_3)$ is the radius-vector of the center of mass: $\boldsymbol{\omega}(p_1, p_2, p_3)$ is the angular velocity vector of the gyrostat: $\boldsymbol{\gamma}(\gamma_1, \gamma_2, \gamma_3)$ is the unit vector of the stationary axis (vertical) directed vertically upward: $\mathbf{e}(e_1, e_2, e_3)$ is the unit vector of the permanent axis: ω is the projection of the angular velocity vector on the vertical; P is the weight of the gyrostat.

1. Investigation of the cone of permanent axes. The equations of motion of a heavy gyrostat having a stationary point can be written as

$$\frac{dI\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times (I\boldsymbol{\omega} + \mathbf{k}) + P(\mathbf{r} \times \boldsymbol{\gamma}) = 0, \qquad \frac{d\boldsymbol{\gamma}}{dt} = \boldsymbol{\gamma} \times \boldsymbol{\omega} \tag{1.1}$$

when its internal motions are steady.

If ω = const, the permanent axis of rotation is the vertical which passes through the stationary point. Then $\mathbf{e} = \boldsymbol{\gamma}$, $\boldsymbol{\omega} = \omega\mathbf{e}$ and from (1.1) we obtain Equation

$$(\mathbf{e} \times I\mathbf{e})\,\omega^2 + (\mathbf{e} \times \mathbf{k})\,\omega + P\mathbf{r} \times \mathbf{e} = 0 \tag{1.2}$$

We now introduce the notation (1.3)

$$S(e_1, e_2, e_3) = I\mathbf{e}(\mathbf{e} \times \mathbf{r}) = (A_2 - A_3)\, r_1e_2e_3 + (A_3 - A_1) r_2e_1e_3 + (A_1 - A_2)\, r_3e_1e_2$$

$$R(e_1, e_2, e_3) = I\mathbf{e}(\mathbf{e} \times \mathbf{k}) = (A_2 - A_3)\, k_1e_2e_3 + (A_3 - A_1)\, k_2e_1e_3 + (A_1 - A_2)\, k_3e_1e_2$$

$$\Pi(e_1, e_2, e_3) = \mathbf{e}(\mathbf{r} \times \mathbf{k}) = N_1e_1 + N_2e_2 + N_3e_3, \qquad N_i = r_{i+1}k_{i+2} - r_{i+2}k_{i+1}$$

Here and below the subscripts must not exceed 3, and should be diminished by 3 if they do.

In the present coordinates e_i the cone of permanent axes is described by a spherical curve obtained by intersecting a unit sphere with a fourth-order surface,

$$\Phi(e_1, e_2, e_3) = P[S(e_1, e_2, e_3)]^2 + R(e_1, e_2, e_3)\,\Pi(e_1, e_2, e_3) = 0 \quad (1.4)$$

Each point of this curve, for which the vectors

$$\mathbf{a} = \mathbf{e}\times I\mathbf{e}, \qquad \mathbf{b} = \mathbf{e}\times\mathbf{k}, \qquad \mathbf{c} = P\mathbf{r}\times\mathbf{e} \qquad (1.5)$$

are nonzero and noncolinear, is associated with a semiaxis which is directed along the vertical and therefore serves as an axis of permanent rotation with the angular veloeity

$$\omega = P\frac{S(e_1, e_2, e_3)}{R(e_1, e_2, e_3)} = -\frac{\Pi(e_1, e_2, e_3)}{S(e_1, e_2, e_3)} \qquad (1.6)$$

For definiteness, let us suppose that

$$A_1 < A_2 < A_3, \quad r_i > 0, \quad k_i > 0, \quad N_1 > 0,\ N_2 < 0,\ N_3 > 0 \qquad (1.7)$$

and then consider the following auxiliary surfaces in the present coordinates e_i.

The surface $S(e_1, e_2, e_3) = 0$. This surface is a second-degree (Staude [5]) cone. For interior points of the cone we have $S > 0$, and for exterior points $S < 0$.

The surface $R(e_1, e_2, e_3) = 0$. This surface is also a second-degree cone. It is the geometric locus of the permanent axes of a gyrostat moving by inertia [6]. For interior points of this cone $R > 0$, and for exterior points $R < 0$.

The surface $\Pi(e_1, e_2, e_3) = \bar{0}$. This plane is the geometric locus of the permanent axes of a heavy spherical gyrostat [7].

For points lying on the side from which the vector $\mathbf{N}(N_1, N_2, N_3)$ normal to the plane emerges $\Pi > 0$: on the other side of the plane $\Pi < 0$.

The cones R and S have four generatrices in common: the coordinate axes Ox_i and the straight line OF with the direction cosines

$$e_{if} = \frac{1}{n}(A_{i+2} - A_{i+1})N_{i+1}N_{i+2}, \qquad n = \pm\left[\sum_{i=1}^{3}(A_{i+1} - A_{i+2})^2 N_{i+2}^2 N_{i+1}^2\right]^{1/2} \qquad (1.8)$$

The plane Π intersects the cone S along the straight line OG, $G(r_1, r_2, r_3)$, the cone R along the straight line OK, $K(k_1, k_2, k_3)$, and both cones simultaneously along the straight line OF.

The lines of intersection of the surfaces S, R, Π, Φ with a unit sphere will be denoted by $\sigma, \rho, \pi, \varphi$, respectively,. The points of intersection of the semiaxes Ox_i, OG, OK, OF with the sphere will be denoted by x_i, g, k, f, and the diametrically opposite points by $-x_i$, $-g$, $-k$, $-f$.

The following statements are valid for the surface Φ and the spherical curve φ.

1. The surface Φ passes through the straight lines Ox_i, OG, and OF, so that the curve φ passes through the points $\pm x_i$, $\pm g$ and $\pm f$ of the sphere.

2. The surface Φ is tangent to the cone R along the coordinate axes. Since the coordinate axes are the lines of intersection of the cones R and S, the surface Φ passes from one side of the cone S to the other side along these axes. Hence, the spherical line φ is tangent to the line ρ and intersects the line σ at the points $\pm x_i$ (Fig. 1).

3. The surface Φ is tangent to the plane Π along the straight line OG, so that the line φ is tangent to the line π at the points $\pm g$.

4. The points of the straight line OF are singular points of the surface Φ in the sense that the normal to the surface Φ is not defined. The surface Φ can intersect itself

along the straight line OF.

5. By virtue of the indicated regions of positiveness and negativeness of the auxiliary surfaces, Equation (1.4) implies that on one side of the plane Π the surface Φ lies entirely on the interior of the cone R, while on the other side of the plane Π the surface Φ lies entirely on the exterior of the cone R. Passage from one side of the plane to the other is effected along the straight line OF.

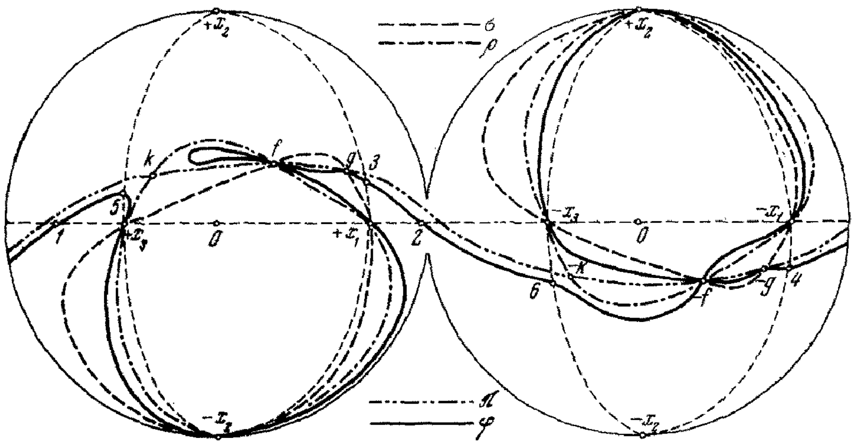

Fig. 1

6. If the quantities k_i tend to zero while remaining proportional to their initial values in such a way that the cone R and the plane Π are not altered, then only the arcs $(x_3, -x_2)$, (f, x_1), $(x_2, -x_1)$, $(-g, -f, -x_3)$ of the line φ can approach the line σ, or, more precisely, the arcs $(x_3, -x_2)$, (g, x_1), $(-x_1, x_2)$, $(-g, -x_3)$ of the line σ (Fig. 1). The points of these arcs of the line σ are determined by the permissible semigeneratrices on the Staude cone in the problem on the permanent rotations of a heavy solid [5 and 8]. If $k_i = 0$, the surface Φ becomes a Staude cone, and the cone R and the plane Π vanish.

On the other hand, if the r_i tend to zero (or if the k_i increase without limit) while remaining proportional to their initial values, then the cone S (or R) and the plane Π do not change, and all the arcs of the line φ approach the lines ρ and π. When $r_i = 0$, the cone S and the plane Π vanish, and the surface Φ becomes the cone R.

The shapes of the lines σ, ρ, π and the presumable shape of the line φ are shown in Fig. 1, where the numbers 1, 2, ..., 6 denote the points of intersection of the line φ with the coordinate planes. The permanent semiaxes through the points 1, 2, ..., 6 are associated with the angular velocities

$$\omega_{1,2} = \frac{Pr_2}{k_2}, \qquad \omega_{3,4} = \frac{Pr_3}{k_3}, \qquad \omega_{5,6} = \frac{Pr_1}{k_1} \tag{1.9}$$

respectively.

The singular points of the line φ are those for which the vectors (1.5) are either collinear or one of them is equal to zero. The permanent rotation for these points cannot be determined on the basis of Formula (1.6). The points $\pm x_i$, $\pm f$, $\pm g$ are singular points.

Let us consider the permanent rotations of a gyrostat about a semiaxis corresponding to one of the singular points.

The vector $\mathbf{a} = \mathbf{e} \times I\mathbf{e}$ is equal to zero for each point $\pm x_1$. For example, let the semiaxis Ox_1 be directed upward along the vertical, i.e. let $\mathbf{e} = \mathbf{j}_1$ where $\mathbf{j}_1$ is the unit vector of the axis x_1. Then $I\mathbf{e} = A_1\mathbf{j}_1$ and Equation (1.2) becomes

$$\mathbf{j}_1 \times (\omega\mathbf{k} - P\mathbf{r}) = 0 \tag{1.10}$$

Since $\mathbf{k} \times \mathbf{r} \neq 0$, Equation (1.10) can be valid only if the axis x_1 lies in the plane Π, i.e. if $N_1 = 0$. In this case we obtain

$$\omega = \frac{Pr_2}{k_2} = \frac{Pr_3}{k_3} \tag{1.11}$$

for the angular velocity of the permanent rotation about the axis x_1.

If the axis x_1 does not lie in the plane Π, i.e. if $N_1 \neq 0$, then Equation (1.10) cannot be satisfied and the axis x_1 cannot be a permanent rotation axis. Generally, if the plane Π does not contain any of the principal axes of inertia ($N_i \neq 0$), then these axes cannot be permanent rotation axes. If any one of the principal axes of inertia lies in the plane Π, then permanent rotations with the same angular velocity correspond to both semiaxes of this axis, and the other two principal axes of inertia cannot be permanent rotation axes.

Vectors (1.5) are collinear for semiaxes through the points $\pm f$. The angular velocity of permanent rotations in this case is given by Equation [3 and 7]

$$(A_1 - A_3)(A_3 - A_2)(A_2 - A_1) N_1 N_2 N_3 \omega^2 + nS(k_1, k_2, k_3)\,\omega + \\ + nPR(r_1, r_2, r_3) = 0 \tag{1.12}$$

If $R(r_1, r_2, r_3) = 0$, i.e. if the center of mass of the gyrostat lies on the cone R, then it is easy to verify that the points $\pm f$ coincide with the points $\pm g$ and that the plane Π is tangent to the cone S along the straight line OG and intersects the cone R along the straight lines OG and OK. For the angular velocity of permanent rotation about the semiaxes through the points $\pm g \equiv \pm f$, (1.12) yields

$$\omega' = 0, \qquad \omega'' = -\frac{nS(k_1, k_2, k_3)}{(A_1 - A_3)(A_3 - A_2)(A_2 - A_1) N_1 N_2 N_3} \tag{1.13}$$

Hence, when the center of mass lies on the cone R, the semiaxes through the center of mass are associated (in addition to the equilibrium position $\omega' = 0$) with a permanent rotation with the angular velocity $\omega'' \neq 0$.

2. The case of dynamic symmetry. Let

$$A_1 \neq A_2 = A_3, \qquad r_i \neq 0, \qquad k_i \neq 0, \qquad \mathbf{k} \times \mathbf{r} \neq 0 \tag{2.1}$$

The surface (1.4) in this case breaks down into the plane

$$e_1 = 0 \tag{2.2}$$

and the third-degree surface

$$P(A_2 - A_1)(r_2 e_3 - r_3 e_2)^2 e_1 + (k_2 e_3 - k_3 e_2)\,\Pi(e_1, e_2, e_3) = 0 \tag{2.3}$$

In order to find the line φ of intersection of surface (2.3) with the unit sphere which defines the cone of permanent axes, let us consider the following auxiliary planes : plane (2.2), the plane Π, and the plane

$$k_2 e_3 - k_3 e_2 = 0 \tag{2.4}$$

Let $N_1 \neq 0$. We can verify directly that surface (2.3) passes through the straight lines Ox_1, OG, Of and Oh. The points $\pm f$ and $\pm h$ on the unit sphere have the following coordinates :

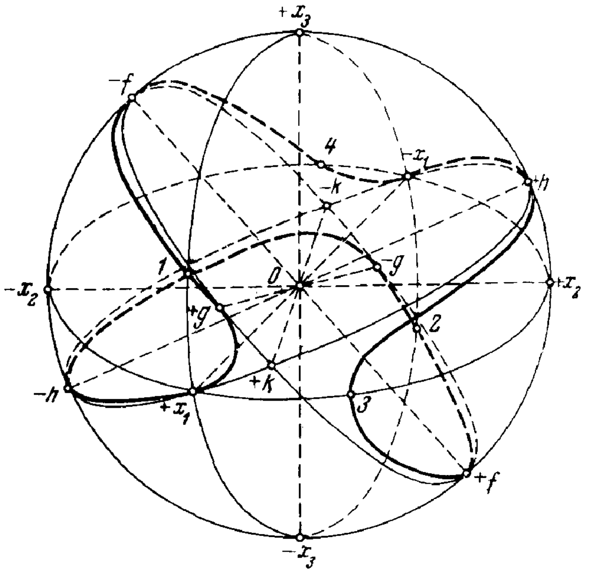

Fig. 2

$$e_{1f} = 0,$$

$$e_{2f} = -\frac{N_3}{\pm\sqrt{N_2^2 + N_3^2}},$$

$$e_{3f} = \frac{N_2}{\pm\sqrt{N_2^2 + N_3^2}} \qquad (2.5)$$

$$e_{1h} = 0,$$

$$e_{2h} = \frac{k_2}{\pm\sqrt{k_2^2 + k_3^2}},$$

$$e_{3h} = \frac{k_3}{\pm\sqrt{k_2^2 + k_3^2}} \qquad (2.6)$$

The straight lines Of and Oh are the lines of intersection of plane (2.2) with the planes Π and (2.4), respectively. The spherical line φ passes through the points $\pm x_1$, $\pm g$, $\pm f$, $\pm h$ of the sphere.

The plane Π is tangent to surface (2.3) along the straight line OG, so that the line φ is tangent at the points $\pm g$ to the line of intersection of the plane Π with the sphere (Fig. 2). Plane (2.4) is tangent to surface (2.3) along the axis x_1, so that the line φ is tangent to the line of intersection of plane (2.4) with the sphere.

From Equation (2.3) of the surface we see that on one side of plane (2.2) (the coordinate plane Ox_2x_3), surface (2.3) is situated entirely in one of the dihedral angles formed by the planes Π and (2.4), while on the other side of plane (2.2) it is situated entirely in the other dihedral angle. Passage from one side of plane (2.2) to the other and from one dihedral angle into the other is effected along the straight lines Of and Oh, respectively.

The points $\pm x_1$, $\pm h$, $\pm f$, $\pm g$ are singular points of the line φ (Fig. 2).

Each nonsingular point of the line φ is associated with a semiaxis which, being directed upward along the vertical, serves as an axis of permanent rotation with the angular velocity

$$\omega = P\frac{r_2e_3 - r_3e_2}{k_2e_3 - k_3e_2} = -\frac{N_1e_1 + N_2e_2 + N_3e_3}{(A_2 - A_1)(r_2e_3 - r_3e_2)e_1} \qquad (2.7)$$

The semiaxes passing through the points $\pm x_1$, $\pm h$ are not associated with permanent rotations. The semiaxes through the points $\pm g$ correspond to gyrostat equilibrium, while the semiaxes through the points $\pm f$ are associated with permanent rotation with the angular velocity

$$\omega = Pr_1/k_1 \qquad (2.8)$$

If $N_1 = 0$, the plane Π merges with plane (2.4).

3. Mechanical Interpretation. Let us suppose that the center of mass does not lie on the axis under consideration, i.e. on the vertical. Following Grammel [9], we shall refer to the plane which passes through the permanent axis (vertical) and the center of mass of the gyrostat as the "vertical central plane" (Fig. 3). This plane and the vectors $\mathbf{K} = \omega I\mathbf{e}$ (the moment of momenta of the gyrostat considered as a solid), $\mathbf{k}$ (the gyrostatic moment), $\mathbf{L} = P\mathbf{e}\times\mathbf{r}$ (the gravitational moment), $\mathbf{M} = \omega^2 I\mathbf{e}\times\mathbf{e}$ (the gyroscopic moment due to the rotation of the gyrostat considered as a solid), and $\mathbf{m} = -\omega\mathbf{e}\times\mathbf{k}$ (the gyroscopic moment due to the internal motions of the gyrostat) remain stationary with respect to the solid portion of the gyrostat and rotate with the angular velocity ω about the vertical. The vectors $\mathbf{L}$, $\mathbf{M}$ and $\mathbf{m}$ lie in the horizontal plane. Let us consider the behavior of these vectors with changing ω. The vector $\mathbf{L}$ remains constant. The vector $\mathbf{M}$ retains its direction, while its magnitude varies as the square of ω. The vector $\mathbf{m}$ retains its direction and varies in magnitude in proportion to ω. When the sign of ω is changed (i.e., when the direction of rotation is altered), only the direction of the vector $\mathbf{m}$ is reversed.

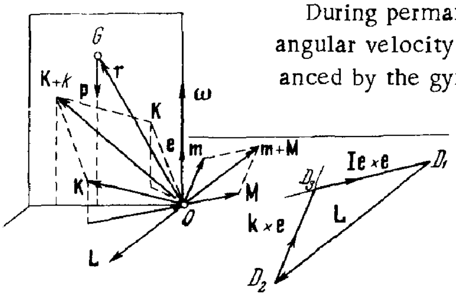

Fig. 3

During permanent rotation of the gyrostat with a constant angular velocity, the gravitational moment $\mathbf{L}$ is counterbalanced by the gyroscopic moments $\mathbf{M}$ and $\mathbf{m}$, i.e. the geometric sum of vectors $\mathbf{L}$, $\mathbf{M}$, $\mathbf{m}$ is zero. This relationship among the vectors $\mathbf{L}$, $\mathbf{M}$ and $\mathbf{m}$ is the basis of our mechanical interpretation of the results obtained in Sections 1 and 2.

Since the gravitational moment is counterbalanced by the total gyroscopic moment $\mathbf{M} + \mathbf{m}$, we conclude that the moment of momenta $\mathbf{K} + \mathbf{k}$ of the gyrostat lies in the vertical central plane.

a) Let us consider the case when the vectors $\mathbf{L}$, $\mathbf{M}$, $\mathbf{m}$ are not collinear. This is possible if the gyrostatic moment $\mathbf{k}$ (and therefore the moment $\mathbf{K}$) do not lie in the vertical central plane, i.e. if

$$\mathbf{e}(\mathbf{r}\times\mathbf{k}) \neq 0 \tag{3.1}$$

This means that the permanent axis does not lie in the plane Π. From the end points D_1 and D_2 of the vector $\mathbf{L}$ (Fig. 3) we construct straight lines parallel to the vectors $\mathbf{k}\times\mathbf{e}$ and $I\mathbf{e}\times\mathbf{e}$ to obtain the triangle $D_1D_2D_3$. From the behavior of the vector $\mathbf{m}$ with changing angular velocity ω we conclude that there exists a unique value of ω for which $\omega\mathbf{k}\times\mathbf{e} = \mathbf{D}_2\mathbf{D}_3$. If it is also the case that $\omega^2 I\mathbf{e}\times\mathbf{e} = \mathbf{D}_3\mathbf{D}_1$, for this same value of ω, then the axis under consideration can be a permanent rotation axis, since the sum of the vectors $\mathbf{L}$, $\mathbf{M}$ and $\mathbf{m}$ is zero,. If, on the other hand, $\omega^2 I\mathbf{e}\times\mathbf{e} \neq \mathbf{D}_3\mathbf{D}_1$ the axis cannot be an axis of permanent rotation. We therefore draw the following conclusion. When the center of mass does not lie on the axis (vertical) and the gyrostatic moment does not lie in the central plane, the angular velocity of this rotation is unique provided the given axis can serve as an axis of permanent rotation.

During permanent rotation the gravitational moment is counterbalanced by the sum $\mathbf{M} + \mathbf{m}$ of the gyroscopic moments, and this is possible only if the radius vector of the center of mass is perpendicular to this sum, i.e. if $(\mathbf{M} + \mathbf{m})\,\mathbf{r} = 0$. Bearing in mind that $\boldsymbol{\omega} = \omega\mathbf{e}$, we find from the latter relation that

$$\omega^2[I\mathbf{e}(\mathbf{e}\times\mathbf{r})] + \omega[\mathbf{e}(\mathbf{r}\times\mathbf{k})] = 0$$

The case $\omega = 0$ is possible only if the center of mass lies on the vertical. In our case $\omega \neq 0$, so that

$$\omega I\mathbf{e}(\mathbf{e}\times\mathbf{r}) + \mathbf{e}(\mathbf{r}\times\mathbf{k}) = 0 \tag{3.2}$$

From Equation (3, 2) we see that permanent rotation is possible if

$$Ie\,(e\times r) = s(e_1, e_2, e_3) \neq 0$$

i. e. if the axis under consideration is not the generatrix of the cone S. Equation (3, 2) then yields

$$\omega = -\frac{e\,(r\times k)}{Ie\,(e\times r)} \tag{3.3}$$

During permanent rotation the moment obtained as the geometric sum of the gravitational moment L and the gyroscopic moment m of the internal motions is counterbalanced by the gyroscopic moment M of the gyrostat considered as a solid .

This is possible only if the vector k is perpendicular to the sum L + m , i. e. if (L + m)k = 0 . Bearing in mind that $\omega = \omega e, \omega \neq 0$, from the latter expression we find that

$$\omega Ie\,(e\times k) - PIe\,(e\times r) = 0 \tag{3.4}$$

and permanent rotation is possible if

$$Ie\,(e\times k) = R\,(e_1, e_2, e_3) \neq 0$$

i. e. the axis under consideration is not the generatrix of the cone R. In this case from (3, 4) we have

$$\omega = P\frac{Ie\,(e\times r)}{Ie\,(e\times k)} \tag{3.5}$$

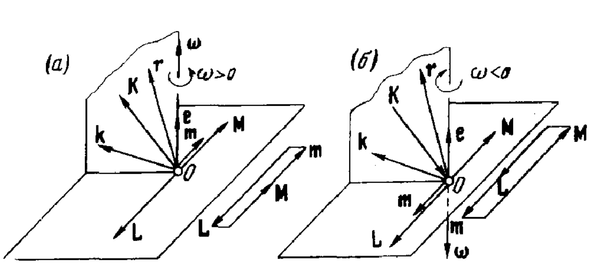

Fig. 4

Expressions (3, 3) and (3, 5) define the angular velocity of permanent rotation about the same axis, and from the uniqueness of ω we conclude that

$$\omega = P\frac{Ie\,(e\times r)}{Ie\,(e\times k)} = -\frac{e\,(r\times k)}{Ie\,(e\times r)} \tag{3.6}$$

From (3, 6) we obtain Equation (1, 4) which defines the position of the permanent axis relative to the solid portion of the gyrostat (in the system $Ox_1x_2x_3$). From the possible positions of the permanent axis defined by Equation (1, 4) we eliminate those for which the permanent axis is the generatrix of the cones R and S and the positions for which the permanent axis passes through the center of mass.

b) Let us consider the case where the vectors L , M , m are collinear. This is possible only if the gyrostatic moment k (and hence the vector K) lies in the vertical central plane, i. e., if

$$e\,(r\times k) = 0 \tag{3.7}$$

This means that the permanent axis lies in the plane Π.

In order to find the angular velocity of permanent rotation and the position of the permanent axis with respect to the solid portion of the gyrostat, let us consider Relations (3, 2) and (3, 4), which are also valid in the case under consideration provided G does not lie on the vertical. Expression (3, 2) in Condition (3, 7) becomes

$$\omega Ie\,(e\times r) = 0$$

so that an angular velocity of permanent rotation $\omega \neq 0$ exists only if

$$Ie\,(e\times r) = S\,(e_1, e_2, e_3) = 0 \tag{3.8}$$

i. e. the axis of permanent rotation must be the generatirix of the cone S. Bearing in mind (3, 8), we find in a similar way from (3, 4) that

$$I\mathbf{e}\,(\mathbf{e}\times\mathbf{k}) = R\ (e_1, e_2, e_3) = 0$$

i.e. the axis of permanent rotation must be the generatrix of the cone R. This brings us to the following conclusion. When the center of mass does not lie on the vertical and the gyrostatic moment lies in the vertical central plane, permanent rotation is possible only if the axis directed along the vertical is the common generatrix of the cones S and R.

Let us consider the case where the common generatrix of the cones R and S, which is not a principal axis of inertia of the gyrostat, is directed upward along the vertical. The vertical divides the central vertical plane into two half-planes. If we look upon them from the end point of the vector $\mathbf{M}$, one will be the right-hand and the other the left-hand half-plane. Let the semiaxis with respect to which the center of mass lies in the right-hand central half-plane be directed upward from the two half-planes of the common generatrix of the cones (Fig. 4). During rotation of the gyrostat in one direction about the axis under consideration (Case a), the vectors $\mathbf{m}$ and $\mathbf{M}$ are directed opposite to the vector $\mathbf{L}$ and there exists a unique (for this direction of rotation) angular velocity ω' for which the geometric sum of these vectors is zero. If we now reverse the direction of rotation of the gyrostat about the same axis (Case b), the vector $\mathbf{m}$ is reversed. Since the vector $\mathbf{L}$ is of constant magnitude independent of ω, it follows that the magnitude of the vector $\mathbf{m}$ is proportional to ω and the magnitude of the vector $\mathbf{M}$ is proportional to the square of ω, it follows that there exists a unique (for this direction of rotation) angular velocity ω'' for which the collinear vectors $\mathbf{L}$, $\mathbf{m}$, $\mathbf{M}$ add up to zero. Thus, the semiaxis under consideration can be the axis of two permanent rotations with angular velocities of different magnitude and sign.

Let the semiaxis relative to which the center of mass lies in the left-hand central half-plane be directed upward from the two semiaxes of the common generatrix of the cones R and S. During rotation of the gyrostat in one direction about this semiaxis, the vectors $\mathbf{L}$, $\mathbf{M}$ and $\mathbf{m}$ are similarly directed and permanent rotation is impossible since there is no angular velocity at which the sum of these vectors is zero. If the gyrostat is made to rotate in the opposite direction about the same semiaxis, the vector $\mathbf{m}$ is directed in the opposite direction, and depending on the magnitudes of the vectors $\mathbf{L}$, $I\mathbf{e}\times\mathbf{e}$, $\mathbf{k}\times\mathbf{e}$ permanent rotation is either impossible or possible. Analysis of these possibilities reduces to determining the existence of real solutions of Equation (1.12).

Let us consider the case where the generatrix of the cones R and S coincides with one of the principal axes of inertia and is directed along the vertical. For example, let the axis x_1 be directed along the vertical. The moment of momenta of the gyrostat considered as a solid is $\mathbf{K} = A_1\omega\mathbf{j}_1$, so that the gyroscopic moment $\mathbf{M}$ is zero. Permanent motion is then possible if the gravitational moment $\mathbf{L}$ is counterbalanced by the gyroscopic moment $\mathbf{m}$ of the internal motions. If the gyrostatic moment is not collinear with the radius vector of the center of mass $(\mathbf{k}\times\mathbf{r}\neq 0)$ and if the gyrostatic moment does not lie in the vertical central plane, the axis x_1 cannot be an axis of permanent rotation since the vectors $\mathbf{L}$ and $\mathbf{m}$ are not collinear and their geometric sum always differs from zero. Generally, if the plane defined by the vectors $\mathbf{r}$ and $\mathbf{k}$ (the plane Π) does not pass through some of the axes x_1, we find that the principal axes of inertia cannot be axes of permanent rotation. When the axis x_1 is directed upward, if the gyrostatic moment lies in the principal central plane (i.e. if the plane Π passes through the axes x_1), the vectors $\mathbf{L}$ and $\mathbf{m}$ are collinear and there exists a unique angular velocity $\omega \neq 0$ for which the sum of these vectors is zero. If the gyrostatic moment is collinear with the radius vector of the center of mass $(\mathbf{k} \times \mathbf{r} = 0)$, then the vectors $\mathbf{L}$ and $\mathbf{m}$ are always collinear and each of the principal axes of inertia can be an axis of permanent rotation.

4. Stability of permanent rotations. Projecting Equations (1.1) on the axes of the system $Ox_1x_2x_3$, we obtain the differential equations of motion of the gyrostat. The particular solution of these equations which corresponds to permanent rotations of the gyrostat can be written as

$$p_i = \omega e_i, \qquad \gamma_i = e_i \tag{4.1}$$

Here the constants e_i satisfy Equation (1.4), and the constant ω is defined in accordance with (1.6). By (4.1), the equations of motion become

$$(A_{i+1} - A_i)\,\omega^2 e_{i+1} e_i + \omega (k_{i+1} e_i - k_i e_{i+1}) + P (r_i e_{i+1} - r_{i+1} e_i) = 0 \tag{4.2}$$

which are identically satisfied by the values of e_i and ω when one of the generatrices of the cone of permanent axes is directed along the vertical.

Let us assume that motion (4.1) is unperturbed and investigate its stability relative to the variables p_i, γ_i. Denoting the variations of these variables by ξ_i, η_i, respectively, and taking account of (4.2), we obtain the equations of perturbed motion which admit of the following first integrals:

$$\begin{gathered} V_1 = \sum_{i=1}^{3} (A_i \xi_i^2 + 2\omega e_i \xi_i + 2 P r_i \eta_i) = \text{const} \\ V_2 = \sum_{i=1}^{3} (A_i \xi_i \eta_i + \omega A_i e_i \eta_i + A_i e_i \xi_i + k_i \eta_i) = \text{const} \\ V_3 = \sum_{i=1}^{3} (\eta_i^2 + 2 e_i \eta_i) = 0 \end{gathered} \tag{4.3}$$

As was done by Rumiantsev [8] in examining the stability of permanent rotations of a solid, we shall make use of integrals (4.3) to construct a Liapunov function of the form

$$V = V_1 - 2\omega V_2 + \lambda V_3 + \frac{\mu}{4} V_3^2 \tag{4.4}$$

where μ is an arbitrary constant, and the constant λ, by virtue of (4.2), is chosen (assuming that $e_i \neq 0$) in the form

$$\lambda = A_i \omega^2 + \frac{k_i \omega - P r_i}{e_i} \qquad (i = 1, 2, 3)$$

Function (4.4) is an integral of fixed sign, and by virtue of Liapunov's stability theorem the unperturbed motion is stable if the inequalities

$$\begin{gathered} \mu e_1^2 + \frac{k_1\omega - P r_1}{e_1} > 0 \\ \mu \left(e_1^2 \frac{k_2 \omega - P r_2}{e_2} + e_2^2 \frac{k_1 \omega - P r_1}{e_1} \right) + \frac{k_1 \omega - P r_1}{e_1} \frac{k_2 \omega - P r_2}{e_2} > 0 \\ \mu \sum_{i=1}^{3} e_i^2 \frac{k_{i+1}\omega - P r_{i+1}}{e_{i+1}} \frac{k_{i+2}\omega - P r_{i+2}}{e_{i+2}} + \prod_{i=1}^{3} \frac{k_i \omega - P r_i}{e_i} > 0 \end{gathered} \tag{4.5}$$

are fulfilled.

If the permanent axis lies in one of the principal planes of inertia, then one of the

e_i, e.g. e_1, is equal to zero, and, taking $\mu = 0$ and

$$\lambda = A_2\omega^2 + \frac{k_2\omega - Pr_2}{e_2} = A_3\omega^2 + \frac{k_3\omega - Pr_3}{e_3}$$

with the aid of function (4.4) we find that the sufficient conditions for the stability of permanent motion are of the form

$$(A_2 - A_1)\,\omega^2 + \frac{k_2\omega - Pr_2}{e_2} > 0, \quad \frac{k_2\omega - Pr_2}{e_2} > 0, \quad \frac{k_3\omega - Pr_3}{e_3} > 0 \qquad (4.6)$$

The problem as to the stability of permanent rotations of a heavy gyrostat about the principal axes of inertia (when two of the e_i are simultaneously equal to zero) is considered in [4].

Let us determine some of the stability domains on the cone of permanent axes using sufficient stability conditions (4.5) and (4.6). We shall consider the stability of permanent rotations about the cone generatrices passing through points 1 - 5 of the line φ (Fig. 1). Let us suppose that the angular velocities ω given by (1.6) and corresponding to the interior points of the arc 1 - 5 lie between values (1.9) of the angular velocities ω_1 and ω_5 of the end points of this arc, i.e. that $\omega_1 < \omega < \omega_5$. By (1.7) we have $\omega_{3,4} < \omega_{1,2} < \omega_{5,6}$ and find that the inequalities

$$k_1\omega - Pr_1 = k_1(\omega - \omega_5) < 0, \quad k_2\omega - Pr_2 = k_2(\omega - \omega_1) > 0$$
$$k_3\omega - Pr_3 = k_3(\omega - \omega_3) > 0$$

for the interior points ($e_1 < 0$, $e_2 > 0$, $e_3 > 0$) of the arc 1 - 5.

Hence we conclude that the sufficient stability conditions (4.5) are fulfilled for $\mu = 0$. From inequality (4.6) we conclude that the permanent rotations about the end points 5 and 1 of the arc 1 - 5 of the line φ are stable.

Similarly we find that permanent motions about the semiaxes passing through the points of the arcs $(x_3, 5)$, $(1, 4)$, $(4, -g)$, $(-g, -f)$, $(-f, -x_3)$ of the line φ are stable.

For the interior points of the arc $(x_3, -x_2)$ of the line φ (Fig. 1) we have $e_1 < 0$, $e_2 < 0$, $e_3 > 0$, and, by (1.6), $\omega < 0$. In addition, $\omega \to \infty$ as the point (e_1, e_2, e_3) approaches certain of the end points x_3 or $-x_2$ of the arc under consideration. The first and second inequalities of (4.6) are fulfilled for each interior point of the arc $(x_3, -x_2)$ for any value $\mu > 0$. The third inequality of (4.5) can be satisfied by some positive value of μ provided that

$$\sum_{i=1}^{3} e_i \frac{k_{i+1}\omega - Pr_{i+1}}{e_{i+1}} \frac{k_{i+2}\omega - Pr_{i+2}}{e_{i+2}} > 0 \qquad (4.7)$$

Since within a sufficiently small neighborhood of the point x_3 the values of e_3 are close to $+1$ while the values of e_1 and e_2 are close to zero, it is clear that inequality (4.7) is fulfilled on some portion of the arc $(x_3, -x_2)$ which extends from the point x_3. In other words, the permanent rotations corresponding to semiaxes passing through points of this portion of the arc $(x_3, -x_2)$ are stable. Similarly, we find that permanent rotations are also stable about semiaxes passing through points of some portion of the arc $(x_2, -x_3)$ extending from the point $-x_3$;

If the gyrostat is dynamically symmetrical and if

$$A_2 = A_3 > A_1, \quad N_1 < 0, \quad N_2 > 0, \; N_3 > 0, \quad r_i > 0$$

then permanent rotations about semiaxes through interior points of the arcs $(3, -f)$,

$(-f, 2)$, $(2, -g)$, $(-g, -h)$, $(h, 3)$ of the line φ (Fig. 2) are stable.

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